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Axially Symmetric Rotating Body Consisting of a Perfect Fluid

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An alternative method of obtaining the equilibrium configurations of a rotating body consisting of a perfect fluid is outlined. Basically, the method involves recasting the gravitational hydrodynamic equations into a set of partial differential equations of first order in the radial direction such that a center-outward integration can be performed. Specifically, with suitable initial conditions at the origin of an r, θ grid, a numerical integration is performed outward along a number of selected θ-rays, with the required θ derivatives at each step being determined numerically from the values of the functions on the different rays. Applicable to both Newtonian and relativistic formulations, the technique is similar to that often used to obtain equilibrium configurations in spherically symmetric models.

1. INTRODUCTION

In the period 1700–1900, the basic mathematical and physical formalism required to tackle the problem of obtaining the equilibrium configurations of a rotating fluid body was built up through the contributions of such scientists as Newton, Clairout, Legendre, Laplace, Jacobi, Poisson, and Poincaré. In the period following 1900, the classical techniques were refined and, with the aid of the computer, new ones were devised. More recently, various researchers have adapted some of these Newtonian methods to the relativistic case (Hartle and Thorne, 1968, Borner and Cohen, 1973; Hartle and Munn, 1975; Sarkisyan and Chubaryan, 1977).

In this paper, another method for determining the equilibrium configurations of an axially symmetric and rotating body is outlined. Basically, the approach involves recasting the gravitational hydrodynamic equations into a set of partial differential equations of first order in the radial direction

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such that a center-outward integration can be performed to obtain a unique equilibrium configuration. Applicable to both Newtonian and relativistic formulations, the technique is a natural extension of the method used to obtain equilibrium configurations in spherically symmetric models (Schwarzschild, 1958).

Basically, the problem under investigation consists in finding distributions of angular velocity, pressure, density, and the gravitational field (both inside and outside the body) that are consistent with the equations of the gravitational-hydrodynamic theory in question. Since there are usually many such compatible distributions, it is necessary to specify subsidiary and boundary conditions in order to determine a unique configuration.

In the modeling of stars, astrophysicists are also concerned with various thermodynamic properties, such as convection, circulation, temperature distribution, and stability against oscillations. Such considerations lie outside the scope of this paper. That is, this paper is concerned specifically with the problem of determining an equilibrium configuration influenced only by mechanical properties, such as pressure, density, velocity, energy density, and, of course, the gravitational field. To that end, for the Newtonian model, a set of partial differential equations that are amenable to a method of center-outward integration for obtaining an equilibrium configuration is derived. Then, a similar set of partial differential equations is obtained for the relativistic model.

2. THE NEWTONIAN CASE

2.1. A Set of Equations for the Newtonian Model

Now, in a classical theory, a rotating body consisting of a perfect fluid can be described by seven Newtonian equations involving the following Eulerian variables:

$$(\varepsilon, \rho, p, v, V) \tag{1}$$

with ε, the internal energy density per unit mass; ρ the rest-mass density; p the pressure; v the velocity of the fluid; and V the gravitational potential. Letting a comma and a semicolon denote the ordinary partial and covariant derivatives, respectively, we can write these seven Newtonian equations, where G is the gravitational constant, as follows:

Equation of continuity:

$$\rho_{,m} v^m + \rho v^a{}_{;a} = 0 \tag{2}$$

Equations of motion:

$$v_{a;m} v^m = -V_{,a} - P_{,a}/\rho \tag{3}$$

Newtonian law of gravitation:

$$g^{\mu\nu}V_{;\mu\nu} = 4\pi G\rho \tag{4}$$

Conservation of energy:

$$\varepsilon_{,\mu} = (p/\rho^2)\rho_{,\mu} \tag{5}$$

Equation of state:

$$p = p(\rho) \tag{6}$$

Note that, in order that they be determinable easily in all coordinate systems, these Newtonian equations have been written in terms of a Newtonian metric and a fourth component of the velocity with respect to time. For our purposes, the only admissible coordinate systems will be those wherein $v^t = 1$. Note that, with regard to equation (4) in a Newtonian theory, the time derivatives of the gravitational potential V are considered to be zero.

2.2. Imposing Axial Symmetry on the Newtonian Model

Now, in spherical coordinates, assuming axial symmetry, $\varepsilon(r, \theta)$, $r(r, \theta)$, $p(r, \theta)$, $V(r, \theta)$, $v^3(r, \theta)$, and $v^t = 1$; the Newtonian metric is

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r\sin\theta \end{bmatrix} \tag{7}$$

From equation (7), it follows that the only surviving connection terms are

$$\begin{aligned} \Gamma^1_{22} &= -r; \qquad \Gamma^2_{12} = 1/r; \qquad \Gamma^1_{33} = -r\sin\theta \\ \Gamma^2_{33} &= -\sin\theta\cos\theta; \qquad \Gamma^3_{13} = 1/r; \qquad \Gamma^3_{23} = \cot\theta \end{aligned} \tag{8}$$

Now the equation of continuity becomes a trivial identity of the form $0 = 0$. One equation of motion also becomes a trivial identity, with the other two yielding:

$$-(1/r)v_3v^3 = -V_{,1} - p_{,1}/\rho \tag{9}$$

$$-(\cot\theta)v_3v^3 = -V_{,2} - p_{,2}/\rho \tag{10}$$

Furthermore, the equation of gravitation becomes

$$V_{,11} = -V_{,22}/r^2 - 2V_{,1}/r - (\cot\theta)V_{,2}/r^2 + 4\pi G\rho \tag{11}$$

2.3. Obtaining the Center-Outward Form of the Newtonian Equations

Now, to rearrange this set of Newtonian equations into a form suitable for center-outward integration, let

$$E = dp/d\rho, \qquad Y = \ln \rho \tag{12}$$

Then, since $v_3 = g_{33}v^3$, the two equations of motion become

$$-r \sin^2 \theta (v^3)^2 = -V_{,1} - EY_{,1} \tag{13}$$

$$-\cot \theta \sin^2 \theta r^2 (v^3)^2 = -V_{,2} - EY_{,2} \tag{14}$$

whence, taking the derivative of equation (13) with respect to θ and the derivative of equation (14) with respect to r and subtracting, we find

$$v^3_{,1} = \frac{\tan \theta v^3_{,2}}{r} + \frac{E_{,1}Y_{,2} - E_{,2}Y_{,1}}{2r \sin \theta \cos \theta} \tag{15}$$

Now, assuming an equation of state of the form $p = K\rho^{\Phi}$, where K and Φ are suitable constants, the outward-integrable set of equations describing the Newtonian model can be written as

$$p = p(\rho) \tag{16}$$

$$V_{,1} = Z \tag{17}$$

$$Z_{,1} = -(\cot \theta V_{,2} + V_{,22})/r^2 - 2Z/r + 4\pi G e^Y \tag{18}$$

$$Y_{,1} = [(v^3)^2 r \sin^2 \theta - Z]/E \tag{19}$$

$$\varepsilon = p/[\rho(\Phi - 1)] \tag{20}$$

$$v^3_{,1} = \tan \theta\, v^3_{,2}/r \tag{21}$$

2.4. Obtaining Equilibrium Configurations in the Newtonian Case

Equations (13)–(21) can be integrated (numerically) from the center of the rotating body outward along selected θ directions (i.e., along selected rays). In the process of center-outward integration, the values of the θ derivatives are obtained numerically at each step of the integration from the values of the functions on the different rays. However, it must be noted that, for the center and the z axis, in order to carry out center-outward integration, equations (16)–(21) must be rewritten to remove indeterminate expressions of the form 0/0. This rewriting can be done with the aid of l'Hôspital's rule of elementary calculus.

In order to start the integration, it is necessary to specify suitable initial conditions at the center of the rotating body. By use of axial symmetry we find that the Z of equation (17) is zero at the origin. Furthermore, since the gravitational potential V is determined only to within an arbitrary constant, V can be assigned the value of zero at the origin. Now, upon examining equation (18), it is found that, at the origin,

$$Z_{,1} = 4\pi G e^{Y}/3 - (\cot\theta\, Z_{,12} + Z_{,122})/6 \tag{22}$$

Thus, a gravitational potential V must be chosen so as to satisfy equation (22). An examination of equation (22) yields that, at the origin, the second radial derivative of potential V can take the form

$$V_{,11} = (\tfrac{4}{3})\,\pi\rho + \lambda(3\cos^2\theta - 1) \tag{23}$$

where λ is an arbitrary constant.

What about the specification of the angular velocity v^3 at the origin? Fortunately, equation (21) can be integrated to give

$$v^3 = \sum_{n=0}^{\infty} v^3_{2n} r^{2n} \sin^{2n}\theta \tag{24}$$

from which it is clear that it is impossible to specify all the information regarding the rotation of the body from the values at the origin of v^3 and $v^3_{,1}$, $v^3_{,11}$, etc., without going to infinite-order derivatives in the radial direction. However, in dimensionless units (where the radius of the body along the equatorial plane is assigned the value 1), the terms of the summation sequence in equation (24) are of decreasing significance. Consequently, for practical calculations only a finite number of the constants v_{2n} need to be specified. Thus, it is clear that equation (24) can be used to replace equation (21) in the outward-integration set. With the aid of equation (23) the equilibrium equations (13)–(14) can be integrated to give

$$V = \sum_{n=0}^{\infty} \frac{v^3_{2n} r^{2n+2}}{2n+2} \sin^{2n+2}\theta + \frac{E_c - E}{\Phi - 1} \tag{25}$$

2.5. Juncture Conditions and the Surface of the Rotating Fluid Body

At the boundary of a rotating fluid, where pressure and density vanish, the functions of Newtonian and relativistic gravitational theories are nonanalytic. Does this situation at the surface of the body introduce restrictions on the values that the variables of the theory can take at the center of the star?

Since in Newtonian physics the essential requirements of the juncture conditions are that the gravitational potential V and its first derivative be continuous across the surface, it follows that the center-outward integration can be extended beyond the surface of the body to determine the gravitational field outside the body without any restrictions at the origin.

2.6. Boundary Conditions at Infinity

The constant λ of equation (23) is not determined by the boundary conditions at the surface of the star. Is the constant λ determined by the boundary conditions at infinity? The arguments and the numerical results of the following sections show that λ is restricted, but not determined by the boundary conditions at infinity. And since λ is not determined uniquely, it follows that *for each angular velocity distribution for which the star has an outer boundary there exists an infinite number of equilibrium configurations for each central density.*

The question now arises as to whether or not an outward integration will yield a gravitational potential that approaches a constant value as $r \to \infty$. It seems reasonable to assume that, at least for some sequences of the coefficients v_{2n} (i.e., $v_{2n}=0$ for all $n >$ some integer N), the values of pressure and density become less than any arbitrarily chosen small values for sufficiently large r. That is, the body does have an outer limit. In this case outside the body the equatons reduce to

$$p=0, \qquad \rho=0, \qquad v^3=0 \tag{26}$$

$$V_{,1}=Z \tag{27}$$

$$Z_{,1}=-(\cot\theta\, V_{,2}+V_{,22})/r^2-2Z/r \tag{28}$$

Now, if the terms involving $V_{,2}$ and $V_{,22}$ become insignificant compared to the term $-2Z/r$, for sufficiently large r, equation (28) becomes

$$Z_{,1}=-2Z/r \tag{29}$$

which can be analytically integrated to yield

$$V_{,1}\propto 1/r^2 \tag{30}$$

Naturally if, for sufficiently large r, equation (28) is not approximated by equation (29), one would not expect the solution to exhibit the desired behavior that the potentials at infinity be well behaved.

3. THE RELATIVISTIC CASE

3.1. A Set of Equations for the Relativistic Model

Relativistically, a rotating body consisting of a perfect fluid can be described in terms of 13 relativistic equations involving 17 unknowns:

$$(\varepsilon, \rho, p, u^a, g_{\mu\nu}) \tag{31}$$

with ε the internal energy density; ρ the proper rest density; p the pressure; u^a the 4-velocity of the fluid; and $g_{\mu\nu}$ the covariant metric coefficients.

Upon imposition of four coordinate conditions, a set of 17 equations in 17 unknowns can be obtained. These relativistic gravitational hydrodynamic equations are:

Equation of continuity:

$$(\rho u^{\nu})_{;\nu} = 0 \tag{32}$$

Equations of motion:

$$T^{\mu\nu}_{;\nu} = 0 \tag{33}$$

Einstein field equations:

$$R_{\mu\nu} = 8\pi(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T) \tag{34}$$

Normalization equation:

$$g_{\mu\nu}u^{\mu}u^{\nu} = -1 \tag{35}$$

Equation of state:

$$p = p(\rho) \tag{36}$$

3.2. Imposing Axial Symmetry on the Relativistic Model

Assume that the geometry of a stationary rotating fluid body has:

(i) A killing vector field ξ_t that is everywhere timelike and has unit length at the origin.

(ii) A killing vector field ξ_θ that is everywhere spacelike and with closed orbits, which in the infinitesimal neighborhood of the radial origin is orthogonal to ξ_ϕ and whose length is given by $r \sin\theta$.

(iii) A 4-velocity of the fluid such that

$$u = u^t \xi_t + u^{\phi} \xi_{\phi} \tag{37}$$

(iv) An equatorial plane of symmetry.

It can then be argued that a unique spherical coordinate system ($x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = t$) can be found that satisfies condition (ii) and in which the metric can be given the form

$$ds^2 = dr^2 + Q^2\, d\theta^2 + Q^2 e^{2B} \sin^2\theta\, d\phi^2 - e^{-2X}\, dt^2 + 2W \sin^2\theta\, d\phi\, dt \tag{38}$$

where Q, B, X, and W are functions of r and θ only.

An examination of these equations reveals that, as in the Newtonian case, the equation of continuity has become a trivial identity. Furthermore, if the stress-energy tensor is expressed as

$$T_{\mu\nu} = Hu_{\nu}u_{\mu} + pg_{\mu\nu}, \qquad \text{where} \quad H = 1 + \varepsilon + p/\rho \tag{39}$$

the two surviving equations of motion are

$$u_3 u^3_{,1} + u_4 u^4_{,1} = -EY_{,1}, \qquad u_3 u^3_{,2} + u_4 u^4_{,2} = -EY_{,2} \tag{40}$$

$$E = (1/H)\, dp/d\rho, \qquad Y = \ln \rho \tag{41}$$

Now, since only six of the field equations are nontrivial and since two of the Bianchi identities are nontrivial, there are only four independent field equations. Hence, with the imposition of axial symmetry, it appears that the result is an indeterminate system of eight equations in nine unknowns (i.e., two equations of motion, four field equations, one normalization equation, and one equation of state). However, another equation, namely,

$$D\varepsilon/DT = (p/\rho)\, D\rho/DT \tag{42}$$

(which is derived from the equations of motion and the normalization equation) survives in the form

$$D\varepsilon/D\rho = p/\rho^2 \tag{43}$$

So, there are in fact nine equations in nine unknowns.

3.3. A Set of First-Order Partial Differential Equations for the Relativistic Model

Can the above system of equations be put in a form suitable for outward integration? Now, the Einstein field equations are linear in the terms involving second-order derivatives. Hence, they can be solved for the four variables $Q_{,11}$, $B_{,11}$, $X_{,11}$, and $W_{,11}$. Along with some help from the normalization equation, equations (40) and (41) can be solved for $u^3_{,1}$ and $u^4_{,1}$. Thus, it is a fairly straightforward (although somewhat lengthy) procedure to express the equations of the relativistic model as a set of first-order partial differential equations similar to those of equations (16)-(21).

3.4. Obtaining Equilibrium Configurations in the Relativistic Case

In the relativistic case, as in the Newtonian, the investigator has the task of choosing appropriate initial conditions at the center of the rotating fluid. Fortunately, the investigator is free to demand that the metric have a Minkowskian form at the center, namely that

$$ds^2 = dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - dt^2 \tag{44}$$

Hence, Q must vary as r at the origin. Thus, $Q_{,1} = 1$, etc. Naturally, only the derivatives to the first order can be specified in this manner.

As in the Newtonian case, second-order derivatives of the form $A_{,12}$, $A_{,22}$, and $A_{,122}$ (etc.) on the metric coefficients must be chosen to satisfy their respective differential equations at the origin, *whence, if the relativistic case allows nonzero values for these derivatives* (and the Newtonian case strongly suggests that it will), *it follows that there will be families of configurations.*

Now, how should u^3 be specified at the origin? Unfortunately, the differential equation for $u^3_{,1}$ cannot be integrated as is possible in the Newtonian case. Nevertheless, from the correspondence between Newtonian and relativistic theories, it follows that in a sufficiently small region around the origin the expansion of equation (23) is valid. Thus, the equilibrium configurations of both the Newtonian and relativistic models are parametrized by the specification of the central density ρ_c and by constants specifying the nature of the gravitational potential(s) at the center of the body and the central angular velocity $u^3_c(\rho)$ at some minute central radius.

3.5. Juncture Conditions and the Boundary Conditions in the Relativistic Model

In general relativity, the question arises of how to extend the integration outside the body. Here, the integration outside the body can be accomplished by the use of the juncture conditions of Bonnor and Vickers (1981). Indeed, when the same coordinates are used inside and outside the body, the juncture conditions always reduce to the simple requirement that the metric coefficients and their first derivatives be continuous across the boundary of the rotating body.

As in the Newtonian case, one is concerned also with boundary conditions at infinity on the gravitational potentials (i.e., metric coefficients). Does the solution approach a Minkowskian metric? Given a potential distribution on any given sphere outside the body, it follows that one can specify the exterior field in terms of Legendre functions. Since these outside Legendre functions will depend on inverse powers of the radius, it is clear that the exterior field will indeed satisfy the appropriate boundary conditions at infinity.

4. NUMERICAL STUDIES

The center-outward integration method of obtaining equilibrium configurations was applied to the simple case of uniformly rotating polytropes. More specifically, a comparative study using a Tandy 1000 with microsoft basic was carried out with respect to the work of James (1964).

At this point, it should be noted that, although the outward-integration approach appeared to be conceptually sound, there is still a major concern of numerical instability introduced when the θ derivatives calculated at one step are fed back into the equations for the next integration step.

In this investigation, I therefore used three different approaches to examining numerical instability resulting from the calculation of the θ derivatives (i.e., to study the sensitivity of the proposed method to the calculation of the θ derivatives), namely:

1. The STARMOD model: least square fitting to the form $A \sin^2 \theta + B \sin^4 \theta$, where A and B are determined at each integration step by the method of least squares.

2. The STARTAYR model: least square fitting to Taylor expansions to the second order (and smoothed over a given number of rays around each ray) in terms of θ.

3. The STARCOMB model: least square fitting to the form of step 1 on the rays near the pole and to the form $A \cos^2 \theta + B \cos^4 \theta$ on the rays near the equator.

Again, according to the theoretical underpinnings of the outward-integration approach to finding the equilibrium configurations investigated by James (1964), there exists a family of equilibrium solutions. Consequently, in order to make the desired comparisons, it was necessary to select the element of that family that specifically corresponds to James' solutions. In particular, it was necessary to specify the value of the constant λ. By trial and error it was found that if 2.5 times the angular velocity u^3 squared (where angular velocity is specified in dimensionless units formed using the speed of light and the radius of the star) is used for λ James data could be reproduced within reasonable limits. This value of λ was used for all the simulations presented.

All the numerical data presented are in terms of dimensionless units as defined by James (1964). Table I provides the data obtained from the three models described above and the data provided by James when a polytropic index of 2, 15 θ rays, and a James rotation of $A = 0.005375$ were used. The comparison is done at a point near the largest central velocity that terminates, since at this point the differences between the solutions is largest.

The differences in the data in Table I confirm that the calculations of the θ derivatives are numerically a significant part of the method. Nevertheless, the three approaches outlined above exhibit similar behaviors, with the STARCOMB model consistently yielding results closest to that of James (1964).

At this point, it should be noted that no method has been found for determining the absolute error in any of the methods. Consequently, in this

Table I. Comparison of the Various Models[a]

	James	Starmod	Starcomb	Startayl
R_p	4.05671	4.289	4.292	4.395
R_e	6.12214	6.209	6.145	5.141
M	2.6501	2.644	2.644	2.626
E_g	0.006312	0.00849	0.00853	0.0515

[a] R_p is the radius of the pole at termination, R_e is the radius of the equator at termination, M is the mass of the star at termination, and E_g is the effective equatorial gravity.

Table II. Comparison of Selected rotations over the Range[a]

V_c	R_p	R_e	M	E_g
0.00	4.352	4.352	2.411	0.1272
	4.359	4.359	2.412	0.1275
0.25	4.339	4.383	2.420	0.1239
	4.352	4.384	2.421	0.1245
1.00	4.299	4.480	2.447	0.1136
	4.317	4.481	2.448	0.1142
1.50	4.272	4.552	2.466	0.1062
	4.308	4.549	2.467	0.1071
2.00	4.245	4.633	2.486	0.0984
	4.295	4.623	2.486	0.0995
2.50	4.218	4.723	2.506	0.0901
	4.290	4.717	2.506	0.0910
3.00	4.190	4.826	2.528	0.0812
	4.285	4.823	2.528	0.0819
3.50	4.153	4.947	2.551	0.0713
	4.277	4.946	2.550	0.0721
4.00	4.135	5.095	2.575	0.0603
	4.277	5.100	2.573	0.0609
4.50	4.106	5.287	2.600	0.0474
	4.280	5.300	2.597	0.0480
5.00	4.078	5.578	2.628	0.0305
	4.290	5.605	2.623	0.0313
5.375	4.057	6.122	2.650	0.0063
	4.292	6.145	2.644	0.0085

[a] See Table I for abbreviations; also, V_c is the central velocity in units as specified by James. The upper number is from James' data; the lower number is from STARMOD data.

study, the error analysis had to be restricted to comparisons with James' data. (For the STARCOMB model, the total numerical error in the solution may be considered as composed of two parts, a Runge-Kutta error and a derivative error. Here, the Runge-Kutta error is of the fifth order in step size h. The θ derivative error may be considered as the error introduced when the values of the θ derivatives are calculated at each step and the error that results from these step errors being integrated over all the remaining steps of the integration. Naturally, if there are many integration steps, a small numerical error early in the integraion could result in considerable error further in the integration.)

Table II provides a comparison over selected rotations between the James data and the data obtained by me from the STARMOD computer program. The data of Table II essentially reproduce the data of the earlier investigations by James. It is therefore reasonable to assume that the outward-integration method described provides another method of determining equilibrium configurations.

5. DISCUSSION

One of the significant findings of this paper is the fact that the outward-integration method does not select an equilibrium configuration from a family of such configurations, but rather details all such families.

Now, the finding of an infinite number of equilibrium configurations appears at first to be in disagreement with the earlier work of James (1964), Ostriker and Bodenheimer (1968), and Ostriker and Mark (1968), in which unique equilibrium configurations were determined. However, a review of the work of James reveals that his method (of finding an equilibrium configuration of a rotating polytrope) consisted in an iterative approach in which he looked for a solution that satisfied the continuity requirements at the surface of the body. The subtle point is that, in James' iterative method, the radius at which this condition is to be satisfied is not known beforehand. Consequently, although his method leads to an equilibrium configuration, there can be, for the same central density, other equilibrium configurations that terminate at different equatorial radii.

In outlining their principal reasons for relating the potential and density by integral relations instead of differential equations, as we have done here, Ostriker and Mark (1968) state: "the boundary conditions are naturally incorporated in the evaluation of the integral, making it unnecessary to go through the tedious procedure of supplementing equation (6) [our equation (4)] by an arbitrary solution to Laplace's equation and determining the unknown constants by matching the potential and the potential gradient across the boundary with an appropriate external solution."

At this point, it must be asked what constitutes an appropriate external solution. For our part, an appropriate external solution is one in which the gravitational potential approaches zero (or a constant) as r approaches infinity and one for which the gravitational potential satisfies Laplace's equation. Now, since the general solution to Laplace's equation is known (in terms of a series involving Legendre functions), it can be seen that there is an infinite number of appropriate external solutions.

It follows, then, that there exist many equilibrium solutions (corresponding to the different external solutions). However, Ostriker and Mark do not say to which external solution their self-consistent-field method converges. Again, it is seen that the iterative nature of the self-consistent-field method, while ensuring that the final solution is an equilibrium configuration, in no way excludes the possibility that, for a given angular velocity distribution, other equilibrium configurations might not also exist. One of the significant aspects of the proposed method is that it does not select an equilibrium configuration from a family of equilibrium configurations, but rather details all such families.

The results of this study, then, are not in disagreement with the work of James, but rather serve to point out a characteristic of James' approach that seems to have been overlooked. Indeed, the numerical studies confirm that, by increasing the central angular velocity until the point is reached at which the effective equatorial gravity at termination is zero, one can construct a sequence of solutions. For central angular velocities beyond the point where the effective gravity at termination is zero, density as a function of the radius from the center of the star follows a rising and falling pattern, the characteristics of which need further investigation.

6. CONCLUSION

Many researchers have studied the problem of determining the equilibrium configuration of a rotating fluid body. In general, the approach taken by these researchers can be grouped into three catgories: (1) those wherein it is assumed that the departure form spherical symmetry is small. Examples of this method are the Clairout–Legendre expansion (Lebovitz, 1970), the Chandrasekhar–Milne expansion (Milne, 1923), and the quasispherical method (Takeda, 1934); (2) those wherein various distributions, such as that of the angular velocity, are assumed *ad hoc* to take a particular mathematical form (Kopal, 1973); and (3) those that involve an iterative approach on trial distributions (Ostriker and Mark, 1968).

The center-outward integration approach discussed here has certain advantages over the previous methods. First, the outward-integration approach is a natural extension of the method used in spherically symmetric models. Second, since the method involves a set of partial differential equations, all the theorems of PDEs are available for analysis. Third, the outward-integration method is equally applicable to Newtonian or relativistic theories.

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